

## Complex Numbers

**Imaginary Number:**  $i = \sqrt{-1}$  and  $i^2 = -1$

**Complex Numbers:** The set of all numbers of the form  $a + bi$  where  $a$  and  $b$  are real numbers.

$a$  is called the **real part** and  $b$  is called the **imaginary part**. If  $b \neq 0$ , then  $a + bi$  is an **imaginary number**. The form  $a + bi$  is called the **standard form** of a complex number.

**Examples:** Determine whether each complex number is real or imaginary and write it in standard form.

- a)  $4i$   
imaginary  
 $0 + 4i$
- b)  $3 - 6i$   
imaginary  
 $3 - 6i$
- c)  $5$   
real  
 $5 + 0i$
- d)  $\frac{i - 3\pi}{4}$   
imaginary  
 $-\frac{3\pi}{4} + \frac{1}{4}i$

**Addition, Subtraction, and Multiplication of Complex Numbers.**

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$

$$(a + bi) - (c + di) = (a - c) + (b - d)i$$

$$(a + bi)(c + di) = (ac - bd) + (bc + ad)i \text{ or use FOIL.}$$

**Examples:** Perform the indicated operations.

$$\begin{aligned} \text{a) } (6 + 2i) + (4 - 3i) &= 6 + 4 + 2i - 3i \\ &= 10 - i \end{aligned}$$

$$\begin{aligned} \text{b) } (7 - 4i) - (-2 + 8i) &= 7 + 2 - 4i - 8i \\ &= 9 - 12i \end{aligned}$$

$$\begin{aligned} \text{c) } (6 + 5i)(8 + 3i) &= \\ 48 + 18i + 40i + 15i^2 &= \\ 48 + 58i + 15(-1) &= \\ \boxed{33 + 58i} \end{aligned}$$

$$\begin{aligned} \text{d) } (1 - i)(4 + i) &= 4 + i - 4i - i^2 \\ &= 4 - 3i - (-1) \\ &= \boxed{5 - 3i} \end{aligned}$$

### Powers of $i$ :

Since  $i^2 = -1$ ,  $i^3 = i^2 \cdot i = -1 \cdot i = -i$ , and  $i^4 = i^2 \cdot i^2 = -1 \cdot -1 = 1$

The first eight powers are listed here:

$$\begin{aligned} i^1 &= i & i^5 &= i \\ i^2 &= -1 & i^6 &= -1 \\ i^3 &= -i & i^7 &= -i \\ i^4 &= 1 & i^8 &= 1 \end{aligned}$$

The powers of  $i$  continue in this pattern.

Exponent $\div 4$	
Remainder	Simplifies To:
0	1
1	$i$
2	-1
3	$-i$

**Examples:** Simplify the power of  $i$ .

$$\begin{aligned} \text{a) } i^{35} &= i^{34} \cdot i \\ &= (i^2)^{17} \cdot i \\ &= (-1)^{17} \cdot i \\ &= \boxed{-i} \end{aligned}$$

$$\begin{aligned} \text{b) } i^{29} &= i^{28} \cdot i \\ &= (i^2)^{14} \cdot i \\ &= (-1)^{14} \cdot i \\ &= \boxed{i} \end{aligned}$$

$$\begin{aligned} \text{c) } i^{98} &= (i^2)^{49} \\ &= (-1)^{49} \\ &= \boxed{-1} \end{aligned}$$

$$\begin{aligned} \text{d) } i^{48} &= (i^2)^{24} \\ &= (-1)^{24} \\ &= \boxed{1} \end{aligned}$$

**Theorem:** If  $a$  and  $b$  are real numbers, then the product of  $a+bi$  and its conjugate  $a-bi$  is the real number  $a^2+b^2$ .  $(a+bi)(a-bi) = (a^2+b^2)$

**Examples:** Find the product of the complex number and its conjugate.

$$\begin{aligned} \text{a) } 3-7i \\ (3-7i)(3+7i) \\ = 3^2 + 7^2 = \boxed{58} \end{aligned}$$

$$\begin{aligned} \text{b) } 2+9i \\ (2+9i)(2-9i) \\ = 2^2 + 9^2 = \boxed{85} \end{aligned}$$

$$\begin{aligned} \text{c) } i \\ i \cdot (-i) \\ = -i^2 = \boxed{1} \end{aligned}$$

**Examples:** Write each quotient in the form  $a+bi$ .

$$\begin{aligned} \text{a) } \frac{6-2i}{3} &= \\ \boxed{2 - \frac{2}{3}i} \end{aligned}$$

$$\begin{aligned} \text{b) } \frac{2}{8+9i} &= \\ \left( \frac{2}{8+9i} \right) \left( \frac{8-9i}{8-9i} \right) &= \\ \frac{16-18i}{8^2+9^2} &= \frac{16-18i}{145} = \\ \boxed{\frac{16}{145} - \frac{18}{145}i} \end{aligned}$$

$$\begin{aligned} \text{c) } \frac{4-5i}{3+2i} &= \left( \frac{4-5i}{3+2i} \right) \left( \frac{3-2i}{3-2i} \right) = \\ \frac{12-8i-15i+10i^2}{3^2+2^2} &= \\ \frac{12-23i-10}{13} &= \\ \frac{2-23i}{13} &= \boxed{\frac{2}{13} - \frac{23}{13}i} \end{aligned}$$

### Roots of Negative Numbers

For any positive real number  $b$ ,  $\sqrt{-b} = i\sqrt{b}$ .

**Examples:** Write each expression in the form  $a+bi$ , where  $a$  and  $b$  are real numbers.

$$\begin{aligned} \text{a) } \sqrt{-5} + \sqrt{-8} &= \\ i\sqrt{5} + i\sqrt{8} &= \\ i\sqrt{5} + 2i\sqrt{2} &= \\ \boxed{(\sqrt{5} + 2\sqrt{2})i} \end{aligned}$$

$$\begin{aligned} \text{b) } \sqrt{-20}(\sqrt{-6} - \sqrt{-4}) &= \\ i\sqrt{20}(i\sqrt{6} - 2i) &= \\ 2i\sqrt{5}(i\sqrt{6} - 2i) &= \\ 2i^2\sqrt{30} - 4i^2\sqrt{5} &= \\ \boxed{-2\sqrt{30} + 4\sqrt{5}} \end{aligned}$$

$$\begin{aligned} \text{c) } \frac{-2+\sqrt{-48}}{2} &= \frac{-2+i\sqrt{48}}{2} \\ &= \frac{-2+4i\sqrt{3}}{2} = \boxed{-1 + 2\sqrt{3}i} \end{aligned}$$

**Example:** Does the complex number  $x = 1+3i\sqrt{2}$  satisfies the equation  $x^2 - 2x + 4 = 0$ ?

$$\begin{aligned} (1+3i\sqrt{2})^2 - 2(1+3i\sqrt{2}) + 4 &\stackrel{?}{=} 0 \\ 1 + 6i\sqrt{2} + (3i\sqrt{2})^2 - 2 - 6i\sqrt{2} + 4 &\stackrel{?}{=} 0 \\ 1 + 9 \cdot i^2 \cdot 2 - 2 + 4 &\stackrel{?}{=} 0 \\ 1 - 18 - 2 + 4 &\stackrel{?}{=} 0 \\ -15 &\neq 0 \end{aligned}$$

**not a solution**

## Trigonometric Form of Complex Numbers

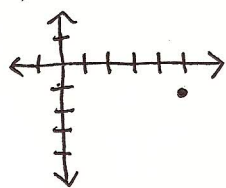
The complex number  $a + bi$  can be thought of as an ordered pair  $(a, b)$ .

We graph it on the **complex plane** where the horizontal axis is called the **real axis** and the vertical axis is called the **imaginary axis**.

**Absolute Value or Modulus:**  $|a + bi| = \sqrt{a^2 + b^2}$ . (The distance between the number and the origin on the complex plane.)

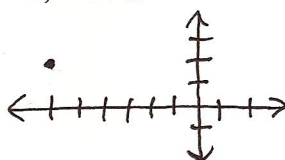
**Examples:** Graph each complex number and find its absolute value.

a)  $5 - i$

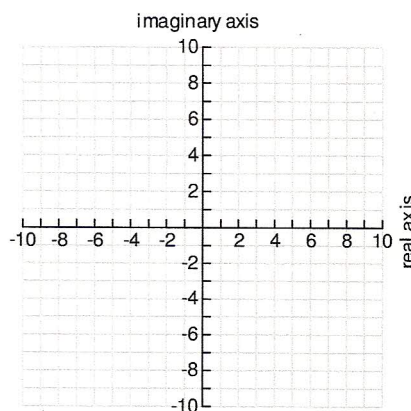


$$|5 - i| = \sqrt{5^2 + (-1)^2} = \sqrt{26}$$

b)  $-6 + 2i$



$$|-6 + 2i| = \sqrt{(-6)^2 + 2^2} = \sqrt{40} = 2\sqrt{10}$$



### Trigonometric Form of a Complex Number

If  $z = a + bi$  is a complex number, then the trigonometric form of  $z$  is

$$z = r(\cos \theta + i \sin \theta), \text{ sometimes abbreviated } z = r \operatorname{cis} \theta,$$

where  $r$  is called the **modulus** and  $\theta$  is called the **argument**, defined as the angle in standard position whose terminal side contains the point  $(a, b)$ .

$$r = \sqrt{a^2 + b^2}$$

$$a = r \cos \theta \text{ and } b = r \sin \theta.$$

We usually use the smallest possible nonnegative angle for  $\theta$ .

**Examples:** Write each complex number in trigonometric form.

a)  $-2\sqrt{3} + 2i$  QII

$$r = \sqrt{(-2\sqrt{3})^2 + 2^2} = 4$$

$$\cos \theta = \frac{-2\sqrt{3}}{4} = -\frac{\sqrt{3}}{2}$$

$$\theta = \frac{5\pi}{6}$$

$$4 \left( \cos \left( \frac{5\pi}{6} \right) + i \sin \left( \frac{5\pi}{6} \right) \right) \text{ or } 4 \operatorname{cis} \left( \frac{5\pi}{6} \right)$$

b)  $5 - 4i$  QIV

$$r = \sqrt{5^2 + (-4)^2} = \sqrt{41}$$

$$\cos \theta = \frac{5}{\sqrt{41}} \quad \sin \theta = \frac{-4}{\sqrt{41}}$$

$$\theta = 321.3^\circ$$

$$\sqrt{41} \left( \cos 321.3^\circ + i \sin 321.3^\circ \right) \text{ or } \sqrt{41} \operatorname{cis} 321.3^\circ$$

**Example:** Write the complex number  $12\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$  in the form  $a+bi$ .

$$12\left(\frac{\sqrt{3}}{2} + i \cdot \frac{1}{2}\right) = \boxed{6\sqrt{3} + 6i}$$

### Product and Quotient of Complex Numbers

If  $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ , and  $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$ , then

$$z_1 z_2 = r_1 r_2 [\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} [\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)]$$

**Examples:** Find the product and quotient using trigonometric form.

$$z_1 = 4\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right), \quad z_2 = 8\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right)$$

a) Find  $z_1 z_2$

$$\begin{aligned} z_1 z_2 &= 4 \cdot 8 (\cos(\frac{\pi}{12} + \frac{\pi}{12}) + i\sin(\frac{\pi}{12} + \frac{\pi}{12})) \\ &= \boxed{32(\cos(\frac{\pi}{6}) + i\sin(\frac{\pi}{6}))} \\ &= 32\left(\frac{\sqrt{3}}{2} + i \cdot \frac{1}{2}\right) = \boxed{16\sqrt{3} + 16i} \end{aligned}$$

b) Find  $\frac{z_1}{z_2} = \frac{4}{8} (\cos(\frac{\pi}{12} - \frac{\pi}{12}) + i\sin(\frac{\pi}{12} - \frac{\pi}{12}))$

$$= \boxed{\frac{1}{2}(\cos 0 + i\sin 0)}$$

$$= \frac{1}{2}(1 + 0i)$$

$$= \boxed{\frac{1}{2}}$$

### Complex Conjugates

The conjugate of  $r(\cos(\theta) + i\sin(\theta))$  is  $r(\cos(-\theta) + i\sin(-\theta))$

**Example:** Find the product of the following and its conjugate:  $6\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$ . Conjugate:

$$6\left(\cos(-\frac{\pi}{3}) + i\sin(-\frac{\pi}{3})\right)$$

$$6\left(\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)\right) \cdot 6\left(\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right) =$$

$$6 \cdot 6 (\cos(\frac{\pi}{3} + (-\frac{\pi}{3})) + i\sin(\frac{\pi}{3} + (-\frac{\pi}{3}))) =$$

$$36(\cos 0 + i\sin 0) =$$

$$36(1 + 0i) = \boxed{36}$$



## Powers and Roots of Complex Numbers

### De Moivre's Theorem

If  $z = r(\cos \theta + i \sin \theta)$  is a complex number and  $n$  is any positive integer, then

$$z^n = r^n (\cos n\theta + i \sin n\theta)$$

### Examples:

a) Simplify  $(1+i)^6$ .

$$1+i \quad r = \sqrt{1^2 + 1^2} = \sqrt{2}$$

$$\text{QI} \quad \cos \theta = \frac{1}{\sqrt{2}} \quad \sin \theta = \frac{1}{\sqrt{2}}$$

$$\theta = \pi/4$$

$$\begin{aligned} & [\sqrt{2} (\cos \pi/4 + i \sin \pi/4)]^6 = \\ & (\sqrt{2})^6 (\cos (6 \cdot \pi/4) + i \sin (6 \cdot \pi/4)) = \\ & 8 (\cos 3\pi/2 + i \sin 3\pi/2) = \\ & 8(0 - i) = \boxed{-8i} \end{aligned}$$

### Roots of a Complex Number

How many square roots does 4 have?

How many square roots does -9 have?

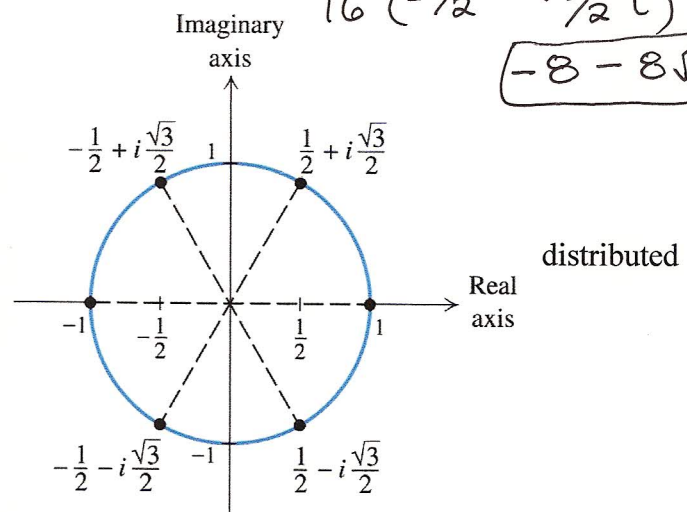
How many sixth roots does 1 have?

It turns out that 1 has 6 sixth roots, and they are evenly around the complex plane.

b) Simplify  $(\sqrt{3}-i)^4$ .  $r = \sqrt{(\sqrt{3})^2 + (-1)^2} = 2$

QIV  $\cos \theta = \frac{\sqrt{3}}{2} \quad \sin \theta = -\frac{1}{2} \quad \theta = \frac{11\pi}{6}$

$$\begin{aligned} & [2 (\cos 11\pi/6 + i \sin 11\pi/6)]^4 = \\ & 2^4 (\cos (4 \cdot 11\pi/6) + i \sin (4 \cdot 11\pi/6)) = \\ & 16 (\cos 22\pi/3 + i \sin 22\pi/3) = \\ & 16 (\cos 4\pi/3 + i \sin 4\pi/3) = \\ & 16 (-\frac{1}{2} - \frac{\sqrt{3}}{2}i) = \\ & \boxed{-8 - 8\sqrt{3}i} \end{aligned}$$



Sixth Roots of Unity

The complex number  $a+bi$  is an  $n$ th root of the complex number  $z$  if  $(a+bi)^n = z$ .

For any positive integer  $n$ , the complex number  $z = r(\cos \theta + i \sin \theta)$  has exactly  $n$  distinct  $n$ th roots given by the expression  $r^{1/n} \left[ \cos \left( \frac{\theta + 360^\circ k}{n} \right) + i \sin \left( \frac{\theta + 360^\circ k}{n} \right) \right]$  for  $k = 0, 1, 2, \dots, n-1$ .

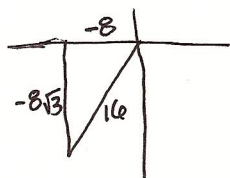
In radians, the roots are given by  $r^{1/n} \left[ \cos \left( \frac{\theta + 2k\pi}{n} \right) + i \sin \left( \frac{\theta + 2k\pi}{n} \right) \right]$  for  $k = 0, 1, 2, \dots, n-1$ .

The first of the  $n$  roots has an argument of  $\frac{\theta}{n}$ , and the other roots are spaced  $\left( \frac{360}{n} \right)^\circ$  apart.

(The circle is divided evenly into  $n$  pieces.)

### Examples:

a) Find all of the fourth roots of the complex number  $-8-8i\sqrt{3}$ .



$$r = \sqrt{(-8)^2 + (-8\sqrt{3})^2} = 16$$

$$\cos \theta = \frac{-8}{16} = -\frac{1}{2} \quad \theta = \frac{4\pi}{3}$$

$$n = 4$$

$$\frac{\theta + 2k\pi}{n} = \frac{\frac{4\pi}{3} + 2k\pi}{4}$$

$$= \frac{\pi}{3} + k\frac{\pi}{2}$$

$$\frac{\pi}{3}, \frac{5\pi}{6}, \frac{4\pi}{3}, \frac{11\pi}{6}$$

$$16^{1/4} (\cos \frac{\pi}{3} + i \sin \frac{\pi}{3})$$

$$= 2 (\frac{1}{2} + \frac{\sqrt{3}}{2} i)$$

$$= \boxed{1 + i\sqrt{3}}$$

$$16^{1/4} (\cos \frac{5\pi}{6} + i \sin \frac{5\pi}{6})$$

$$= 2 (-\frac{\sqrt{3}}{2} + \frac{1}{2} i)$$

$$= \boxed{-\sqrt{3} + i}$$

$$16^{1/4} (\cos \frac{4\pi}{3} + i \sin \frac{4\pi}{3})$$

$$= 2 (-\frac{1}{2} - \frac{\sqrt{3}}{2} i)$$

$$= \boxed{-1 - i\sqrt{3}}$$

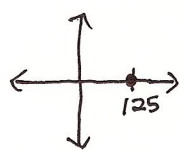
$$16^{1/4} (\cos \frac{11\pi}{6} + i \sin \frac{11\pi}{6})$$

$$= 2 (\frac{\sqrt{3}}{2} - \frac{1}{2} i)$$

$$= \boxed{\sqrt{3} - i}$$

b) Find all the cube roots of 125.

$$125 + 0i$$



$$r = 125 \quad \theta = 0^\circ$$

$$125 (\cos 0^\circ + i \sin 0^\circ)$$

$$n = 3$$

$$\frac{\theta + 360^\circ k}{n} = \frac{0^\circ + 360^\circ k}{3}$$

$$= 120^\circ k$$

$$= 0^\circ, 120^\circ, 240^\circ$$

$$125^{1/3} (\cos 0^\circ + i \sin 0^\circ)$$

$$= 5 (1 + 0i) = \boxed{5}$$

$$125^{1/3} (\cos 120^\circ + i \sin 120^\circ)$$

$$= 5 (-\frac{1}{2} + \frac{\sqrt{3}}{2} i)$$

$$= \boxed{-\frac{5}{2} + \frac{5\sqrt{3}}{2} i}$$

$$125^{1/3} (\cos 240^\circ + i \sin 240^\circ)$$

$$= 5 (-\frac{1}{2} - \frac{\sqrt{3}}{2} i)$$

$$= \boxed{-\frac{5}{2} - \frac{5\sqrt{3}}{2} i}$$

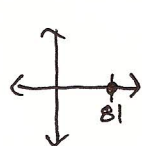
c) Find all complex solutions to  $x^4 - 81 = 0$ .

$$x^4 = 81 \leftarrow 4^{\text{th}} \text{ roots of } 81$$

$$n = 4$$

$$\frac{0^\circ + 360^\circ k}{4} = \frac{360^\circ k}{4} = 90^\circ k$$

$$0^\circ, 90^\circ, 180^\circ, 270^\circ$$



$$r = 81 \quad \theta = 0^\circ$$

$$81 (\cos 0^\circ + i \sin 0^\circ)$$

$$81^{1/4} (\cos 0^\circ + i \sin 0^\circ)$$

$$3 (1 + 0i) = \boxed{3}$$

$$81^{1/4} (\cos 90^\circ + i \sin 90^\circ)$$

$$3 (0 + i) = \boxed{3i}$$

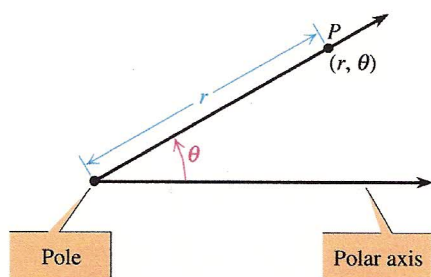
$$81^{1/4} (\cos 180^\circ + i \sin 180^\circ)$$

$$3 (-1 + 0i) = \boxed{-3}$$

$$81^{1/4} (\cos 270^\circ + i \sin 270^\circ)$$

$$3 (0 - i) = \boxed{-3i}$$

## Polar Equations



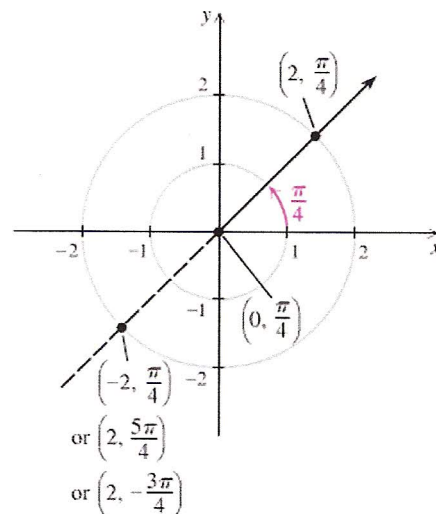
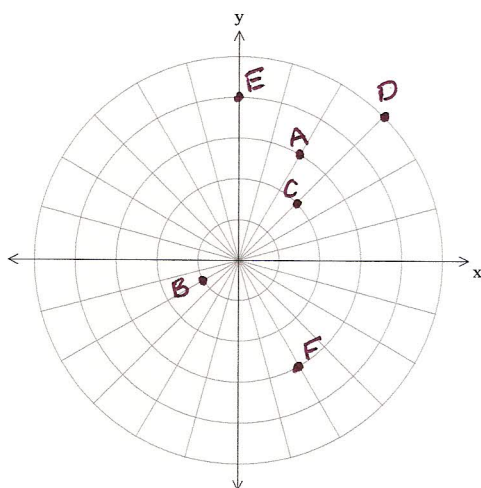
The polar coordinate system is based on a fixed point called the **pole** and a fixed axis called the **polar axis**. Points are represented by ordered pairs in the form  $(r, \theta)$ , where  $r$  is the **directed distance** from the pole and  $\theta$  is an angle whose initial side is the polar axis and whose terminal side contains the point. Typically, we choose the origin as the pole and the positive  $x$ -axis as the polar axis.

\*To graph  $(-r, \theta)$ , you move in the opposite direction you would move to graph  $(r, \theta)$ .

Polar coordinates are not unique. The points  $(-2, \frac{\pi}{4})$ ,  $(2, \frac{5\pi}{4})$ , and  $(2, -\frac{3\pi}{4})$  all name the same point.

**Examples:** Plot the points whose polar coordinates are given.

$A(3, \frac{\pi}{3})$ ,  $B(-1, \frac{\pi}{6})$ ,  $C(2, -\frac{7\pi}{4})$ ,  $D(-5, -\frac{3\pi}{4})$ ,  $E(4, \frac{\pi}{2})$ ,  $F(-3, \frac{2\pi}{3})$



### Polar-Rectangular Conversion Rules

- To convert  $(r, \theta)$  to rectangular coordinates  $(x, y)$ , use  $x = r \cos \theta$  and  $y = r \sin \theta$ .
- To convert  $(x, y)$  to polar coordinates  $(r, \theta)$ , use  $r = \sqrt{x^2 + y^2}$  and any angle  $\theta$  in standard position whose terminal side contains  $(x, y)$ .

**Examples:**

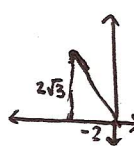
a) Convert  $(3, 45^\circ)$  to rectangular coordinates.

$$x = 3 \cos 45^\circ = 3\left(\frac{\sqrt{2}}{2}\right) = \frac{3\sqrt{2}}{2}$$

$$y = 3 \sin 45^\circ = 3\left(\frac{\sqrt{2}}{2}\right) = \frac{3\sqrt{2}}{2}$$

$$\boxed{\left(\frac{3\sqrt{2}}{2}, \frac{3\sqrt{2}}{2}\right)}$$

b) Convert  $(-2, 2\sqrt{3})$  to polar coordinates.



$$r = \sqrt{(-2)^2 + (2\sqrt{3})^2}$$

$$= \sqrt{4 + 12} = \sqrt{16} = 4$$

$$\cos \theta = -\frac{2}{4} = -\frac{1}{2} \quad \theta = \frac{2\pi}{3}$$

$$\boxed{(4, \frac{2\pi}{3})}$$

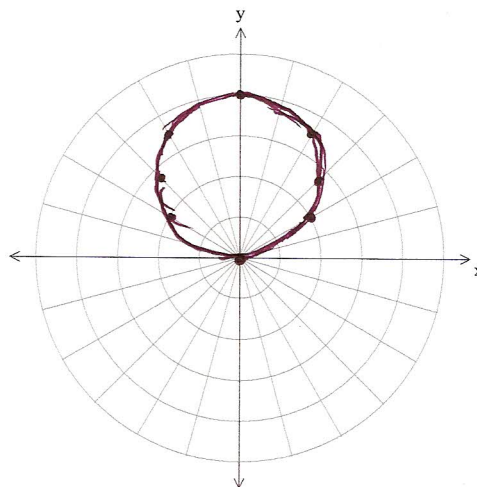


## Graphing Polar Equations

**Examples:** Sketch the graphs of the following:

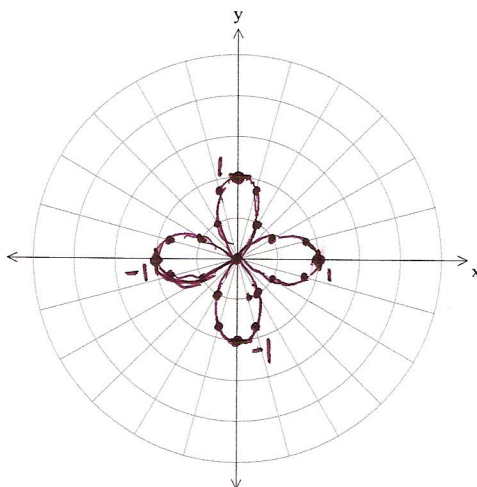
a)  $r = 4 \sin \theta$

$\theta$	$r$
0	$4(0) = 0$
$\pi/6$	$4(1/2) = 2$
$\pi/4$	$4(\sqrt{2}/2) = 2\sqrt{2} \approx 2.8$
$\pi/3$	$4(\sqrt{3}/2) = 2\sqrt{3} \approx 3.5$
$\pi/2$	$4(1) = 4$
$2\pi/3$	$4(\sqrt{3}/2) = 2\sqrt{3} \approx 3.5$
$5\pi/6$	$4(1/2) = 2$
$\pi$	$4(0) = 0$



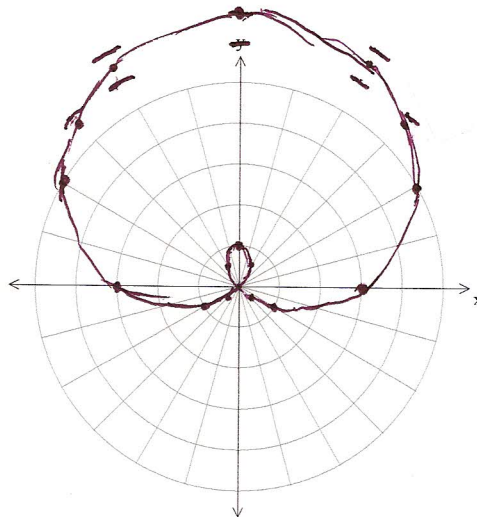
b)  $r = \cos(2\theta)$

$\theta$	$r$	$\theta$	$r$
0	1	$3\pi/4$	0
$\pi/2$	$\sqrt{3}/2 \approx .9$	$\pi$	1
$\pi/6$	$1/2$	$5\pi/4$	0
$\pi/4$	0	$3\pi/2$	-1
$\pi/3$	$-1/2$	$7\pi/4$	0
$5\pi/2$	$-\sqrt{3}/2 \approx -.9$	$2\pi$	1
$\pi/2$	-1		



c)  $r = 3 + 4 \sin \theta$

$\theta$	$r$	$\theta$	$r$
0	3	$7\pi/6$	1
$\pi/6$	5	$5\pi/4$	.2
$\pi/4$	5.8	$4\pi/3$	-.5
$\pi/3$	6.5	$3\pi/2$	-1
$\pi/2$	7	$5\pi/3$	-.5
$2\pi/3$	6.5	$7\pi/4$	.2
$3\pi/4$	5.8	$11\pi/6$	1
$5\pi/6$	5	$2\pi$	3
$\pi$	3		





Use  $x^2 + y^2 = r^2$ ,  $x = r \cos \theta$ ,  $y = r \sin \theta$

**Examples:** Convert equations from polar to rectangular form.

a) Convert  $r = 3 \sin \theta$  to a rectangular equation.

$$r(r) = r(3 \sin \theta)$$

$$r^2 = 3r \sin \theta$$

$$\boxed{x^2 + y^2 = 3y} \leftarrow \text{circle}$$

b) Convert  $r = \frac{4}{1 + \sin \theta}$  to a rectangular equation.

$$r(1 + \sin \theta) = 4$$

$$r + r \sin \theta = 4$$

$$r + y = 4$$

$$r = -y + 4$$

$$\rightarrow r^2 = (-y + 4)^2$$

$$r^2 = y^2 - 8y + 16$$

$$x^2 + y^2 = y^2 - 8y + 16$$

$$\boxed{x^2 = -8y + 16} \leftarrow \text{parabola}$$

**Examples:** Convert equations from rectangular to polar form.

c) Convert  $y = -2x + 5$  to a polar equation.

$$r \sin \theta = -2r \cos \theta + 5$$

$$r \sin \theta + 2r \cos \theta = 5$$

$$r(\sin \theta + 2 \cos \theta) = 5$$

$$\boxed{r = \frac{5}{\sin \theta + 2 \cos \theta}}$$

d) Convert  $x^2 + (y-1)^2 = 1$  to a polar equation.

$$x^2 + y^2 - 2y + 1 = 1$$

$$x^2 + y^2 - 2y = 0$$

$$x^2 + y^2 = 2y$$

$$r^2 = 2r \sin \theta$$

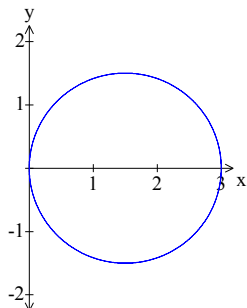
$$\boxed{r = 2 \sin \theta}$$

1) Lines through the origin are of the form  $\theta = \alpha$ .

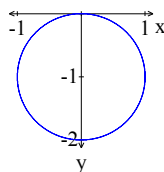
Vertical lines are of the form  $r \cos \theta = a$ .

Horizontal lines are of the form  $r \sin \theta = a$ .

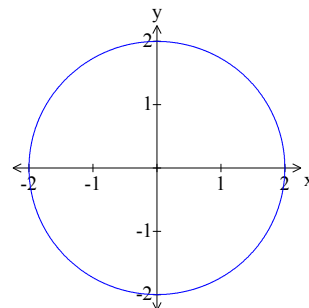
2) Circles come in three forms:  $r = a \cos \theta$ ,  $r = a \sin \theta$ , and  $r = a$ .



$$r = 3 \cos \theta$$



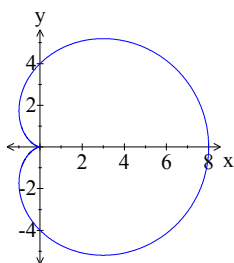
$$r = -2 \sin \theta$$



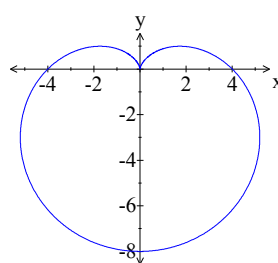
$$r = 2$$

3) Cardioids have the form  $r = a \pm a \cos \theta$  or  $r = a \pm a \sin \theta$ .

Cardioids pass through the pole.



$$r = 4 + 4 \cos \theta$$



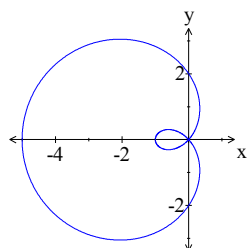
$$r = 4 - 4 \sin \theta$$

4) Limaçons have the form  $r = a \pm b \cos \theta$  or  $r = a \pm b \sin \theta$ .

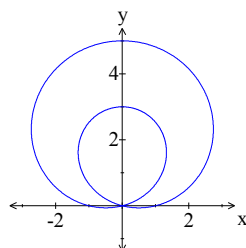
Limaçons have an inner loop if  $0 < a < b$  and have no inner loop if  $0 < b < a$ .

The graph of a limaçon with an inner loop passes through the pole twice.

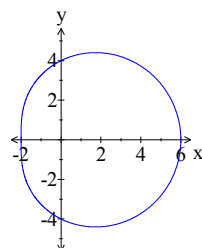
The graph of a limaçon with no inner loop does not pass through the pole.



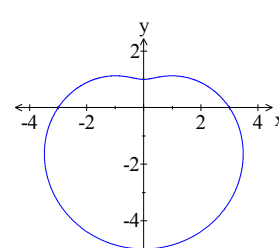
$$r = 2 - 3 \cos \theta$$



$$r = 1 + 4 \sin \theta$$

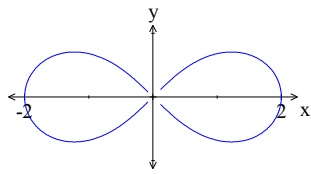


$$r = 4 + 2 \cos \theta$$

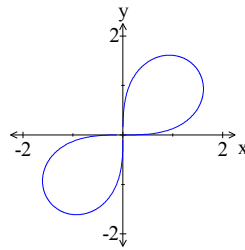


$$r = 3 - 2 \sin \theta$$

5) Lemniscates have the form  $r^2 = a^2 \cos(2\theta)$  or  $r^2 = a^2 \sin(2\theta)$ .



$$r^2 = 4 \cos(2\theta)$$

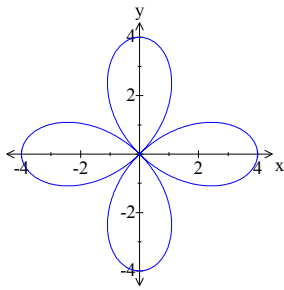


$$r^2 = 4 \sin(2\theta)$$

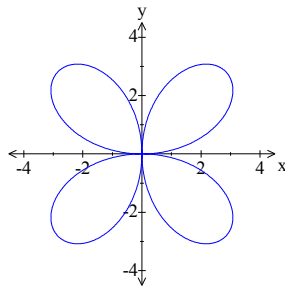
6) Roses have the form  $r = a \cos(n\theta) + b$  and  $r = a \sin(n\theta) + b$ .

If  $n$  is even, there are  $2n$  loops in the rose.

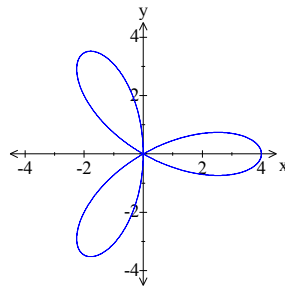
If  $n$  is odd, there are  $n$  loops in the rose.



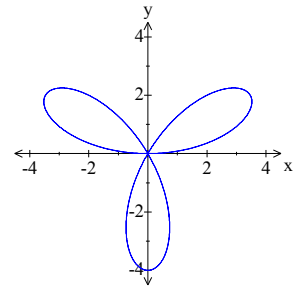
$$r = 4 \cos(2\theta)$$



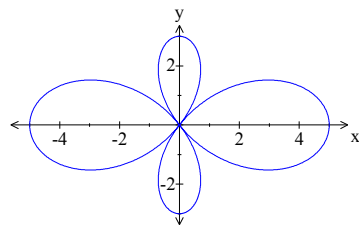
$$r = 4 \sin(2\theta)$$



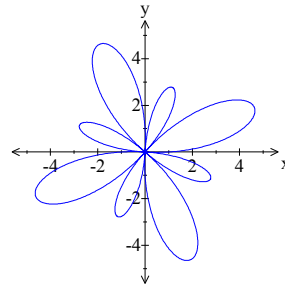
$$r = 4 \cos(3\theta)$$



$$r = 4 \sin(3\theta)$$



$$r = 4 \cos(2\theta) + 1$$



$$r = 4 \sin(4\theta) + 1$$



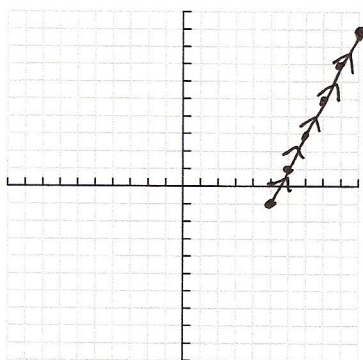
## Parametric Equations

Sometimes, it is convenient to express both  $x$  and  $y$  as functions of a third variable,  $t$ . If  $f(t)$  and  $g(t)$  are both functions of  $t$ , where  $t$  is some interval of real numbers, then the equations  $x = f(t)$  and  $y = g(t)$  are called **parametric equations**. The variable  $t$  is called the **parameter**. If we think of  $t$  as time, then we know when each point of the graph is plotted.

### Graphing Parametric Equations

1. Make a  $t, x, y$  table for the two equations.
2. Plot the ordered pairs of values of  $x$  and  $y$ .
3. Mark the **orientation** of the curve by using arrows to show the direction of the graph.

**Example:** Graph the parametric equations  $x = t + 5$  and  $y = 2t - 1$  for  $t$  in  $[0, 5]$ .



$t$	$x$	$y$
0	5	-1
1	6	1
2	7	3
3	8	5
4	9	7
5	10	9

Graph goes through  $(5, -1)$ ,  $(6, 1)$ ,  $(7, 3)$ ,  $(8, 5)$ ,  $(9, 7)$ , &  $(10, 9)$

### Eliminating the Parameter

1. Set one equation equal to  $t$ .
2. Substitute that equation in for  $t$  in the other equation.
3. Sometimes it is more convenient to use a trigonometric identity to eliminate the parameter.

**Examples:** Eliminate the parameter and identify the graph of the parametric equation.

a)  $x = 4t - 9$ ,  $y = -t + 1$ ,  $-\infty < t < \infty$

$$x + 9 = 4t$$

$$t = \frac{x+9}{4}$$

$$t = \frac{1}{4}x + \frac{9}{4}$$

$$y = -t + 1$$

$$y = -\left(\frac{1}{4}x + \frac{9}{4}\right) + 1$$

$$y = -\frac{1}{4}x - \frac{9}{4} + 1$$

$$\boxed{y = -\frac{1}{4}x - \frac{5}{4}}$$

line

b)  $x = 5 \sin t$ ,  $y = 5 \cos t$ ,  $-\infty < t < \infty$

$$x^2 + y^2 = (5 \sin t)^2 + (5 \cos t)^2$$

$$x^2 + y^2 = 25 \sin^2 t + 25 \cos^2 t$$

$$x^2 + y^2 = 25 (\sin^2 t + \cos^2 t)$$

$$x^2 + y^2 = 25(1)$$

$$\boxed{x^2 + y^2 = 25}$$

circle

## Writing Parametric Equations for Line Segments

1. Write both parametric equations as linear functions:  $x = m_1 t + b_1$ , and  $y = m_2 t + b_2$ .
2. Substitute  $x$  and  $t$  values into the  $x$  equation to create a system of equations you can solve for  $m_1$  and  $b_1$ .
3. Substitute  $y$  and  $t$  values into the  $y$  equation to create a system of equations you can solve for  $m_2$  and  $b_2$ .

### Examples:

Write parametric equations for the line segment starting at  $(1, 2)$  with  $t = 0$  and ending at  $(8, 10)$  with  $t = 1$ .

$$\begin{aligned}
 x &= m_1 t + b_1 & y &= m_2 t + b_2 \\
 t=0, x=1, y=2: & 1 = m_1(0) + b_1 & 2 &= m_2(0) + b_2 \\
 & 1 = b_1 & 2 &= b_2 \\
 t=1, x=8, y=10: & 8 = m_1(1) + 1 & 10 &= m_2(1) + 2 \\
 & m_1 = 7 & m_2 &= 8
 \end{aligned}$$

$$\begin{aligned}
 x &= 7t + 1 \\
 y &= 8t + 2
 \end{aligned}$$

Write parametric equations for the line segment starting at  $(-2, 4)$  with  $t = 3$  and ending at  $(5, -9)$  with  $t = 7$ .

$$\begin{aligned}
 x &= m_1 t + b_1 & y &= m_2 t + b_2 \\
 t=3, x=-2, y=4: & -2 = 3m_1 + b_1 & 4 &= 3m_2 + b_2 \\
 t=7, x=5, y=-9: & 5 = 7m_1 + b_1 & -9 &= 7m_2 + b_2
 \end{aligned}$$

$$\begin{aligned}
 (-2 = 3m_1 + b_1)(-1) \\
 5 = 7m_1 + b_1 & \quad m_1 = 7/4 \\
 2 = -3m_1 - b_1 & \quad -2 = 3(7/4) + b_1 \\
 7 = 4m_1 & \quad b_1 = -29/4
 \end{aligned}$$

$$\begin{aligned}
 (4 = 3m_2 + b_2)(-1) \\
 -9 = 7m_2 + b_2 \\
 -4 = -3m_2 - b_2 \\
 -13 = 4m_2 \\
 m_2 = -13/4
 \end{aligned}$$

$$\begin{aligned}
 x &= \frac{7}{4}t - \frac{29}{4} \\
 y &= -\frac{13}{4}t + \frac{55}{4}
 \end{aligned}$$

### Writing Parametric Equations for a Polar Equation

Use the equations  $x = r \cos \theta$  and  $y = r \sin \theta$ . Replace  $r$  to obtain the parametric equations.

When converting polar equations to parametric equations,  $\theta$  acts as the parameter.

**Example:** Write parametric equations for the polar equation  $r = 3 \cos \theta$ .

$$\begin{aligned}
 x &= r \cos \theta \\
 x &= (3 \cos \theta) \cos \theta \\
 y &= r \sin \theta \\
 y &= (3 \cos \theta) \sin \theta
 \end{aligned}$$

$$\begin{aligned}
 x &= 3 \cos^2 \theta \\
 y &= 3 \cos \theta \sin \theta
 \end{aligned}$$